CONTINUING PROBLEMS

The purpose of the following exercises is to pick up some additional problems.

1. **'Coordinate equation' of a straight line**

   Besides parametric equations for straight lines in space one could also introduce coordinate equations which, however, are impracticable. The following example shows what such a coordinate equation could look like:

   $$(2x + y - 5z + 11)^2 + (x - 3y + z + 2)^2 = 0.$$

   a. Why does this 'coordinate equation' describe a straight line?
      Hint: Think of replacing this equation by an appropriate equation system.
   b. Write down the parametric equation of the straight line.

2. **Parametric equation of a plane**

   We treated the coordinate equations of planes but did not discuss their parametric equations. The following example shows that parametric equations are possible as well:

   $$\vec{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}.$$

   a. Why does this parametric equation describe a plane?
      Hint: Compare with the parametric equation of a straight line.
   b. Write down the coordinate equation of this plane.
   c. What is the general form of the parametric equation of a plane?

3. **Sphere equation**

   Up to now, we solved problems with spheres without explicitly using a sphere equation. The following example shows which form a sphere equation has:

   $$(x-3)^2+(y+1)^2+(z-2)^2 = 9 \;\Leftrightarrow\; x^2+y^2+z^2-6x+2y-4z+5 = 0.$$

   a. Determine the center M and the radius r of the such described sphere.
   b. In which points does the straight line $g : \vec{r} = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ pierce the sphere?

      Hint: Operate with the sphere equation even though its explicit use would not be necessary to solve the problem.
   c. What is the general form of the coordinate equation of a sphere?

4. **Conic sections**

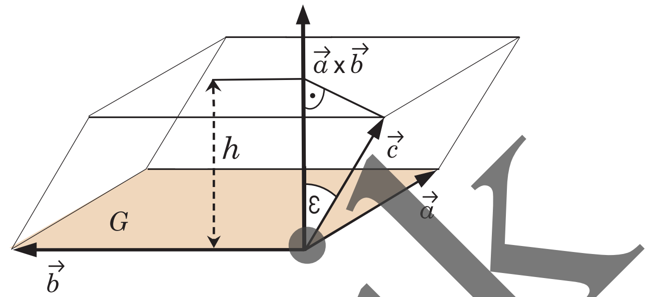
   Let g and h be two different straight lines which are part of one plane. If g rotates around h, we obtain one of the following 'rotational surfaces': In general, a *double cone surface* (on both sides infinite) and in special cases, a *cylinder surface* (also on both sides infinite) if g is parallel to h or a *plane* if g is perpendicular to h. The set of points which occurs by intersecting such a rotational surface with a plane is called a *conic section*. Here, we restrict ourselves to the general case, namely to a double cone surface; this, of course, is defined by an apex, an axis and an opening angle. As an example, consider the double cone surface $\mathcal{D}$ with apex $S = (3/4/2)$, a further axis point $A = (1/3/0)$ and an opening angle $\alpha = 60°$.

   a. What is an equation of $\mathcal{D}$?
   b. What is an equation of the conic section which is obtained by intersecting $\mathcal{D}$ with the xy-plane?
   c. What may be the general form of an equation of a conic section in the xy-plane?
   d. How does the intersection plane have to lie with respect to a double cone surface such that the conic section is an ellipse, a parabola or a hyperbola? Of what kind is the conic section of our example 4b?

5. **Scalar triple product**

We consider a parallelepiped spanned by three non-coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ which are represented from a common initial point. First of all, we want to calculate the volume V of this parallelepiped (base area G and heigth h) in a way that is as elegant as possible.

$\vec{a}$, $\vec{b}$, $\vec{a} \times \vec{b}$ and, according to the figure, also $\vec{a}$, $\vec{b}$, $\vec{c}$ are, in these orders, right-oriented. The angle ε between $\vec{a} \times \vec{b}$ and $\vec{c}$ is therefore either acute or $0°$. Thus, we have:



$$V = Gh = |\vec{a} \times \vec{b}||\vec{c}| \cos \varepsilon$$
$$= (\vec{a} \times \vec{b}) \cdot \vec{c}.$$

By interchanging in this result the vectors $\vec{a}$ and $\vec{b}$ and using calculation laws we obtain:

$$(\vec{b} \times \vec{a}) \cdot \vec{c} = (-(\vec{a} \times \vec{b})) \cdot \vec{c} = -((\vec{a} \times \vec{b}) \cdot \vec{c}) = -V.$$

Thus, the vector terms $(\vec{a} \times \vec{b}) \cdot \vec{c}$ and $(\vec{b} \times \vec{a}) \cdot \vec{c}$ differ only in the sign and their common absolute value is the volume V of the parallelepiped. With the sign itself it possible to define algebraically a sense of *orientation*: The three (non-coplanar) vectors are, in the order of their operands, right-oriented if the vector term is positive and left-oriented if it is negative.

Definition. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three arbitrary vectors. The vector term $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is called a *scalar triple product* and it is shortly written in the form $(\vec{a}, \vec{b}, \vec{c})$.

a. Which relations result from permuting the operands in $(\vec{a}, \vec{b}, \vec{c})$?
b. Again: How can the scalar triple product be geometrically interpreted?
c. When exactly is the scalar triple product equal to 0?
d. Compute the volume of the tetrahedron with the following vertexes: $A = (3/4/4)$, $B = (5/0/5)$, $C = (2/2/-1)$ and $D = (6/2/1)$.

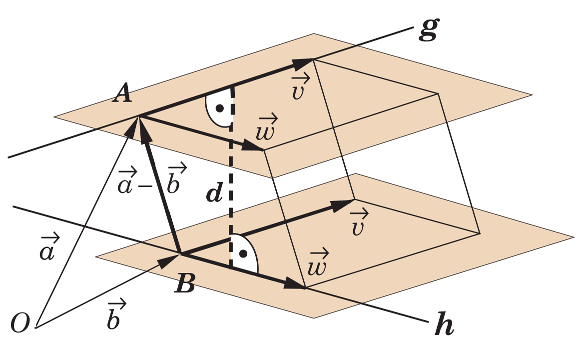
6. **Distance between two skew straight lines**

Two skew straight lines are given: $g : \vec{r} = \vec{a} + t\vec{v}$, $h : \vec{r} = \vec{b} + t\vec{w}$.

The distance d between the straight lines g and h (= the length of the shortest connection line segment between g and h) is equal to the distance between the two uniquely determined parallel planes, one containing g and the other containing h (see also exercise 24, page 47). As shown in the figure below, both planes are spanned by representatives of the vectors $\vec{v}$ and $\vec{w}$ with common initial point, namely A for one plane and B for the other. Hence, the distance problem can be formulated as follows:

The distance d is the height of the parallelepiped which is spanned by representatives of the vectors $\vec{v}$, $\vec{w}$ and $\vec{a} - \vec{b}$ according to the figure. By using the scalar tiple product we get:

$$d = \frac{|(\vec{v}, \vec{w}, \vec{a} - \vec{b})|}{|\vec{v} \times \vec{w}|}$$



$g: \vec{r} = \begin{pmatrix} -2 \\ 7 \\ 6 \end{pmatrix} + t \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$, $h: \vec{r} = \begin{pmatrix} 3 \\ 10 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$. What is the distance d between g and h.