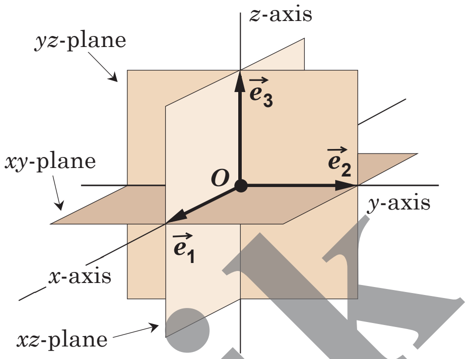
C. VECTORS AND POINTS IN THE COORDINATE SYSTEM

C1. Coordinate system



Definition. An *orthonormal right-handed (Cartesian) coordinate system*, for short a *coordinate system*, consists of a fixed point O, *the origin*, and 3 numbered non-coplanar vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$, the *basis vectors*, which are represented with the common initial point O and fulfill the following three conditions:

- They are *pairwise perpendicular* ('ortho').
- Their norms are each equal to 1, i.e., they are so-called *unit vectors* ('normalised').
- They are arranged in accordance with the 3 spread fingers of the right hand as follows: $\vec{e}_1$ (thumb), $\vec{e}_2$ (forefinger) and $\vec{e}_3$ (middle finger). One says: $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ form, in this order, a *right-handed system*; one also speaks of a right sense of orientation.

The three representatives $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ define three coordinate axis passing through O: the *x-axis*, *y-axis* and *z-axis*. Every two of the three axis specify a coordinate plane: the *xy-plane*, *yz-plane* and *xz-plane*. From now on, we always refer to a coordinate system with the above mentioned properties and notations.

Remark (in view of considerations later on)
Any three non-coplanar vectors (not necessarily basis vectors) in a given order define, according to the above hand rule, a sense of orientation; they are *right-* or *left-oriented*.

C2. Component representation of vectors

According to the theorem in B4 (previous page 6), every vector $\vec{v}$ is a linear combination of the basis vectors: $\vec{v} = x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3$.

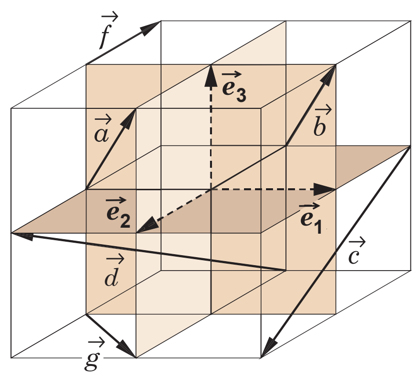
Definition. The coefficients x, y and z of $\vec{v}=x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3$ are called the *components* of $\vec{v}$. One briefly writes $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and speaks of the *component representation* of $\vec{v}$.

(Once more: The components x, y and z describe the expansion of $\vec{v}$ in direction of $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$, respectively.)

In particular, for the basis vectors it yields: $\vec{e}_1 = \begin{pmatrix} \\ \dots\dots \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} \\ \dots\dots \end{pmatrix}$, $\vec{e}_3 = \begin{pmatrix} \\ \dots\dots \end{pmatrix}$.

Example *Vector components*

Specify the components of the vectors represented in the adjoining coordinate system:



$\vec{a} = \begin{pmatrix} \\ \dots\dots \end{pmatrix}$, $\vec{b} = \begin{pmatrix} \\ \dots\dots \end{pmatrix}$, $\vec{c} = \begin{pmatrix} \\ \dots\dots \end{pmatrix}$,

$\vec{d} = \begin{pmatrix} \\ \dots\dots \end{pmatrix}$, $\vec{f} = \begin{pmatrix} \\ \dots\dots \end{pmatrix}$, $\vec{g} = \begin{pmatrix} \\ \dots\dots \end{pmatrix}$.

C3. Calculation with components

Zero vector: $\vec{o} = \begin{pmatrix} \\ \\ \dots\dots \end{pmatrix}$ *Proof:* $\vec{o} = 0\,\vec{e_1} + 0\,\vec{e_2} + 0\,\vec{e_3} = \begin{pmatrix} \\ \\ \dots\dots \end{pmatrix}.$

Opposite vector: $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \Leftrightarrow -\vec{a} = \begin{pmatrix} \\ \\ \dots\dots \end{pmatrix}$

Proof: $\vec{a} = a_1\,\vec{e_1} + a_2\,\vec{e_2} + a_3\,\vec{e_3} \;\Leftrightarrow\; -\vec{a} = -(a_1\,\vec{e_1} + a_2\,\vec{e_2} + a_3\,\vec{e_3})$

By transforming: $-\vec{a} = (-a_1)\,\vec{e_1} + (-a_2)\,\vec{e_2} + (-a_3)\,\vec{e_3} = \begin{pmatrix} \\ \\ \dots\dots \end{pmatrix}.$

Vector norm: $|\vec{a}| = \left|\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right| = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

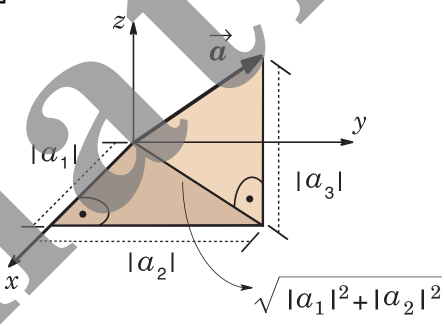
Proof:

$$|\vec{a}| = \sqrt{\left(\sqrt{|a_1|^2 + |a_2|^2}\right)^2 + |a_3|^2}$$

$$= \sqrt{|a_1|^2 + |a_2|^2 + |a_3|^2}$$

$$= \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .$$

(using the Pythagorean theorem twice)



Example *Vector norms*

Calculate: $\left|\begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix}\right| = \dots\dots\dots,\quad \left|\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}\right| = \dots\dots,\quad \left|\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}\right| = \dots\dots.$

Reflect upon: $\left|\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}\right| = \left|\begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}\right| = \left|\begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}\right| = \left|\begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix}\right| = \dots\dots$ etc.

Addition and subtraction: $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \pm \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{pmatrix}$

Proof:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \pm \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = (a_1\,\vec{e_1} + a_2\,\vec{e_2} + a_3\,\vec{e_3}) \pm (b_1\,\vec{e_1} + b_2\,\vec{e_2} + b_3\,\vec{e_3})$$

$$= (a_1 \pm b_1)\,\vec{e_1} + (a_2 \pm b_2)\,\vec{e_2} + (a_3 \pm b_3)\,\vec{e_3} = \begin{pmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ a_3 \pm b_3 \end{pmatrix}.$$

Scalar multiple of a vector: $x \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} x\,a_1 \\ x\,a_2 \\ x\,a_3 \end{pmatrix}$

Proof: $x \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} =$

$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .$

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C4. Coordinate representation of points

By defining the coordinate system the origin O has been introduced. This would not have been necessary for the component representation of vectors since vectors do not depend on position. However, the origin O is essential for the representation of points with coordinates.

There exists a one-to-one correspondence between vectors and points which will be used in the following definition. Let us consider a vector represented by an arrow which starts in the origin O (O is the initial point) and ends in the peak denoted by P (P is the terminal point), then this correspondence can be written as follows:

$$\text{Vector } \overrightarrow{OP} \leftrightarrow \text{ Point } P.$$

Definition. The components x, y, z of $\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are called the *coordinates* of P. One writes $P = (x/y/z)$ and speaks of the *coordinate representation* of P. The vector $\overrightarrow{OP}$ is said to be the *position vector* of the point P.

Remark

Despite the close relation between points and their position vectors, one should never 'add' or 'subtract' points. Furthermore, the difference in notation between the two geometrical objects has to be observed consistently (Coordinates of a point are written in a row, components of a vector in a column).

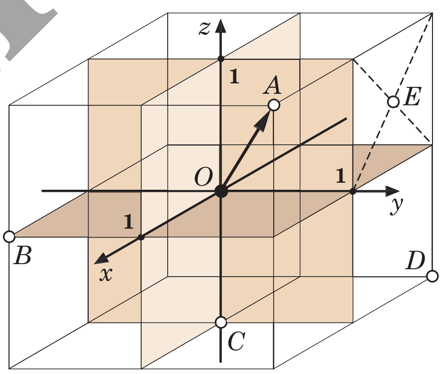
Example *Coordinates of points*

Determine the coordinates of the points given in the adjoining coordinate system (Keep in mind: the position vectors of the points lying on the axis at 1 are the basis vectors):

A = ..., however $\overrightarrow{OA} = \begin{pmatrix} \\ \\ ...... \end{pmatrix}$,

B = ..., C = ...,

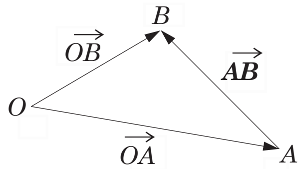
D = ..., E = ...



C5. Exercises

First, we will treat 3 basic problems the solutions of which will be applied continually. In all three problems we use the following two points: $A = (a_1/a_2/a_3)$ and $B = (b_1/b_2/b_3)$.

Basic problem 1: *Connecting vector* $\overrightarrow{AB}$ *of the points* A *and* B



Since $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ one obtains: $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$

Briefly: Coordinates of the terminal point minus the corresponding coordinates of the initial point!

Example *Connecting vector*

Determine $\overrightarrow{AB}$ with $A = (5/-3/1)$ and $B = (7/5/-2)$: $\overrightarrow{AB} = \begin{pmatrix} \\ \\ ..... \end{pmatrix}$.

Basic problem 2: *Line segment length* $\overline{AB}$

Since $\overline{AB} = \left|\overrightarrow{AB}\right|$ one obtains: $\boxed{\overline{AB} = \sqrt{(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2}}$

Note: It is recommended to do the calculations in two steps, namely by first determining the vector $\overrightarrow{AB}$ and then its norm.

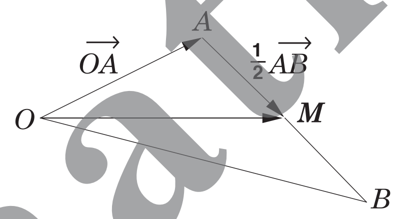
Example *Line segment lenght*

Determine $\overline{AB}$ with $A=(2/-2/1)$ and $B=(3/2/-7)$: $\overline{AB} = \left|\begin{pmatrix} \\ \\ ..... \end{pmatrix}\right| = ...$.

Basic problem 3: *Midpoint* M *of* AB

Since $\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} b_1-a_1 \\ b_2-a_2 \\ b_3-a_3 \end{pmatrix} = \begin{pmatrix} a_1+\frac{1}{2}(b_1-a_1) \\ a_2+\frac{1}{2}(b_2-a_2) \\ a_3+\frac{1}{2}(b_3-a_3) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(a_1+b_1) \\ \frac{1}{2}(a_2+b_2) \\ \frac{1}{2}(a_3+b_3) \end{pmatrix}$
one obtains:

$$\boxed{M = \left(\frac{a_1+b_1}{2} \Big/ \frac{a_2+b_2}{2} \Big/ \frac{a_3+b_3}{2}\right)}$$



Briefly: Arithmetic mean of the corresponding coordinates!

Example *Midpoint of line segment*

Determine the midpoint M of AB with $A=(5/-4/2)$ and $B=(9/2/-2)$:

$M = $ ....

Exercises

The exercises below are meant to be solved on the following empty pages. They can be separated into subproblems corresponding to the three basic problems just discussed. To compute a point, it might be appropriate to determine the corresponding position vector.

Exercise 1 $A=(1/-2/3),\ B=(5/2/-5),\ C=(9/6/-7)$

Determine the midpoints of the sides and the centroid of the triangle ABC.

Exercise 2 $A=(-1/9/4),\ B=(11/3/8),\ C=(t/0/t)$

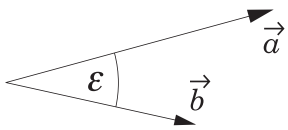
a. Where does the point C lie in the coordinate system for all $t \in \mathbb{R}$?

b. Compute t such that the triangle ABC is isosceles with basis AB.

c. Calculate the area of this triangle.

Exercise 3

$\vec{a} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix},\ \vec{b} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$



Determine a vector in direction of the bisector (bisectrix) of the angle ε.

Exercise 4 $A=(4/4/3),\ B=(2/0/-1)$

In which points does the x-axis pierce the sphere with diameter $\overline{AB}$?

Exercise 5 $A=(4/-1/5),\ B=(0/1/11),\ C=(8/9/5),\ D=(-2/-1/1)$

Show vectorially that the midpoints of the sides of the quadrangle $ABCD$ form the vertexes (corners) of a parallelogram. Is the result generally valid?

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