E. VECTOR PRODUCT

E1. Definition

The scalar product of two vectors is a scalar, the result of the vector product (also known as cross product) of two vectors shall now be a vector. Just as the scalar product is very important in physics, this is also the case for the vector product, for instance, in connection with turning moments (torques), electromagnetic fields and so on. A discussion of physical examples, however, would take us too far afield here. In the context of our analytical geometry the vector product is differently used: It is especially important if one looks for a direction being perpendicular to both of two given directions; furthermore, it serves for the computation of areas.

In the theory of the scalar product we proceeded from a geometrical definition and subsequently deduced the algebraic description, i.e., the component representation. For the vector product the reverse approach will be chosen: First, we will develop the component representation and regard it as the definition of the vector product in order to determine its geometrical properties afterwards.

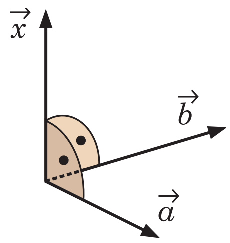
Problem:

$$\underbrace{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}}_{\vec{a}} \times \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}_{\vec{b}} = \underbrace{\begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}}_{\vec{x}}$$

To two given vectors $\vec{a}$ and $\vec{b}$ (operands) we wish to assign a new vector $\vec{x}$ (resulting product vector); the operation symbol is '$\times$' (speak 'cross'). How to define the components of $\vec{x}$?

The product vector shall be perpendicular to the operands:

First, we ask for the direction of the product vector $\vec{x}$ relative to the operands $\vec{a}$ and $\vec{b}$. Physical vector quantities, in connection with the already mentioned turning moments, electromagnetic fields etc., are perpendicular to the generating vector quantities. Therefore, we require:



$$\begin{array}{rl} & \vec{x} \text{ perpendicular } \vec{a} \\ \text{and} & \vec{x} \text{ perpendicular } \vec{b} \end{array} \Leftrightarrow \begin{array}{rl} & \vec{x}\cdot\vec{a} = 0 \\ \text{and} & \vec{x}\cdot\vec{b} = 0. \end{array}$$

Using the component representations $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ as well as $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ an equation system with two equations and three unknowns x_1, x_2 and x_3 is obtained. We eliminate x_3 by applying the addition method:

$$\begin{array}{rcl|l} a_1x_1 + a_2x_2 + a_3x_3 & = & 0 & \cdot b_3 \\ b_1x_1 + b_2x_2 + b_3x_3 & = & 0 & \cdot(-a_3) \end{array}$$

$$\underbrace{(a_1b_3 - a_3b_1)}_{c}\, x_1 + \underbrace{(a_2b_3 - a_3b_2)}_{d}\, x_2 = 0.$$

Thus, one solution looks as follows (presume x_3 and verify by inserting):

$$\begin{array}{lclcl} x_1 & = & d & = & a_2b_3 - a_3b_2 \\ x_2 & = & -c & = & a_3b_1 - a_1b_3 \\ x_3 & & & = & \text{..................} . \end{array}$$

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A selection has to be made:

Also $\vec{x} = k \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$ with $k \in \mathbb{R}$ is a solution because the norm of $\vec{x}$ is not defined with the requirement that $\vec{x}$ has to stand perpendicularly on $\vec{a}$ and $\vec{b}$. Which of the infinitely many solutions should now be chosen for the vector product? Looking at a special case facilitates the decision:

$\vec{a} = \vec{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\vec{b} = \vec{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. The solution with $k = 1$ leads to $\vec{x} = \vec{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Because of this simple and aesthetic result we take the solution with $k = 1$.

Definition. The *vector product* (or cross product) is explained as follows:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

Example *Vector product*

$$\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \\ \\ \ldots\ldots \end{pmatrix}$$

Calculation scheme:

E2. Geometrical properties

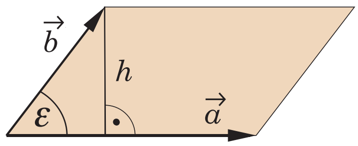
In the following $\vec{a}$ and $\vec{b}$ are two non-collinear vectors.

1. $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$ and $\vec{b}$. (according to the definition above)

2. $|\vec{a} \times \vec{b}|$ is the area of a parallelogram spanned by $\vec{a}$ and $\vec{b}$.

   Proof:

   In order to avoid square roots we regard the square of the area I of the parallelogram. In the case of an obtuse intermediate angle ε the relationship $\sin(180° - \varepsilon) = \sin\varepsilon$ can additionally be used.
   (Note that the step $*$ needs a lot of calculations.)



$$\begin{aligned} I^2 &= |\vec{a}|^2h^2 = |\vec{a}|^2|\vec{b}|^2\sin^2\varepsilon \\ &= |\vec{a}|^2|\vec{b}|^2(1-\cos^2\varepsilon) = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 \\ &= ({a_1}^2 + {a_2}^2 + {a_3}^2)({b_1}^2 + {b_2}^2 + {b_3}^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &\underset{*}{=} ({a_2}^2{b_3}^2 + {a_3}^2{b_2}^2 - 2a_2b_2a_3b_3) + ({a_3}^2{b_1}^2 + {a_1}^2{b_3}^2 - 2a_3b_3a_1b_1) \\ &\quad + ({a_1}^2{b_2}^2 + {a_2}^2{b_1}^2 - 2a_1b_1a_2b_2) = |\vec{a} \times \vec{b}|^2. \end{aligned}$$

3. $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ are, in this order, right-oriented.

   This presupposes that already the basis vectors $\vec{e_1}, \vec{e_2}, \vec{e_3}$ are, in this order, right-oriented, as required by defining the coordinate system.

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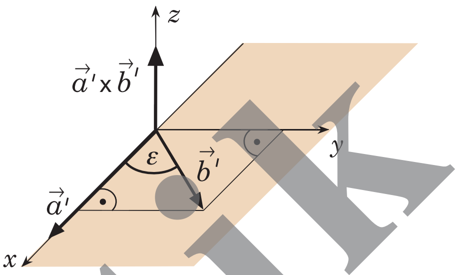
Outline of a Proof:

A pair of representatives $(\vec{a}, \vec{b})$ with common initial point can be transferred into $(\vec{a}', \vec{b}')$ by a proper movement such that the latter pair lies within the shaded xy-half plane as shown in the figure (the representative $\vec{a}'$ has the direction of the positive x-half axis). Considering the vector order corresponding to $\vec{a}, \vec{b}, \vec{a}\times\vec{b}$ of the original vectors during this movement, the sense of orientation does not change since the vector product has a constant norm (= parallelogram area) and continuously varying components (no jumps).

It is $\vec{a}' = \begin{pmatrix} |\vec{a}| \\ 0 \\ 0 \end{pmatrix}$, $\vec{b}' = \begin{pmatrix} |\vec{b}|\cos\varepsilon \\ |\vec{b}|\sin\varepsilon \\ 0 \end{pmatrix}$

and thus $\vec{a}'\times\vec{b}' = \begin{pmatrix} 0 \\ 0 \\ |\vec{a}||\vec{b}|\underbrace{\sin\varepsilon}_{>0,\text{ since } 0^\circ<\varepsilon<180^\circ} \end{pmatrix}$.



$\vec{a}', \vec{b}', \vec{a}'\times\vec{b}'$ and thus $\vec{a}, \vec{b}, \vec{a}\times\vec{b}$ are, in these given orders, right-oriented.

Calculation laws:

$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$. The vector product is said to be *anti-commutative*.

$\vec{a} \times \vec{o} = \vec{o}, \quad \vec{a} \times \vec{a} = \vec{o}$.

In general, the associative law $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$ does not hold which can be easily verified by a counter-example.

Computations with vector products demand some attention!

Most of the calculation laws known from numbers are not valid as soon as vector products are involved. However, since we barely rearrange terms with vector products, no problems will arise.

E3. Area formula and exercises

Because the norm of the vector product can be interpreted as the area of a parallelogram, areas of triangles, and based on these, areas of polygons can be computed. For triangles we immediately get an *area formula* .

Area I of a triangle ABC: $I = \dfrac{|\dots\dots\dots\dots\dots\dots|}{\dots\dots}$

Exercises

From now on the vector product will be applied frequently. At this point, some exercises are to be solved on the following empty pages:

Exercise 1 $A = (1/1/1),\ B = (4/3/3),\ C = (0/-1/3)$

a. Compute the area of the triangle ABC.

b. There are two tetrahedrons $ABCD$ with volume 18 such that the foot of the height outgoing from the vertex D is the center of BC. Determine the possible vertexes D.

Exercise 2 $A = (3/3/2),\ B = (1/1/1),\ C = (-1/2/3)$

Let AB and BC be two edges of a cube.

a. How many cubes with these edges exist, and what is their volume?

b. Compute the other vertexes of the cube where $\overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BF}$, in this order, are left-oriented (BF is the third edge outgoing from vertex B).

Exercise 3 Show: $|k\vec{a} \times \vec{b}| = |\vec{a} \times k\vec{b}| = |k||\vec{a} \times \vec{b}|$.

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